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On the complexity of the traveling umpire problem ☆

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ABSTRACT

The traveling umpire problem (TUP) consists of determining which games will be handled by each one of several umpire crews during a double round-robin tournament. The objective is to minimize the total distance traveled by the umpires, while respecting constraints that include visiting every team at home, and not seeing a team or venue too often. Even small instances of the TUP are very difficult to solve, and several exact and heuristic approaches for it have been proposed in the literature. To this date, however, no formal proof of the TUP's computational complexity exists. We prove that the decision version of the TUP is $\mathcal{NP}$-complete for certain values of its input parameters.

1. Introduction

The traveling umpire problem (TUP) consists of determining which games will be handled by each one of n umpire crews during a double round-robin tournament with $2n$ teams. The objective is to minimize the total distance traveled by the umpires, while respecting constraints that include visiting every team at home, and not seeing a team or venue too often throughout the season. The TUP was created as an abstraction of the real-life umpire scheduling problem faced by Major League Baseball in an attempt to isolate the few features that make the problem difficult to solve (see [1]). Since it was first introduced, several papers have proposed exact and heuristic approaches to tackle the TUP, such as [1–6]. Despite the steady progress in solving progressively larger instances of the problem, empirical evidence shows that the TUP is still a very difficult problem to solve. According to the official TUP benchmark set [7], no instances with more than 10 teams have known optimal solutions.

On the theoretical side, however, the TUP has attracted far less attention. To this date, no formal proof of the TUP's computational complexity exists, and this is the focus of our paper. We are concerned with the decision version of the TUP, as defined below.

Definition 1. Given a double round-robin tournament T with $2n$ teams, the distance d_{ij} between the home venues of any two teams i and j, two non-negative integers $d_1 \le n-1$ and $d_2 \le \lfloor n/2 \rfloor - 1$, and a non-negative number ℓ, the decision version of the TUP consists of determining whether or not there exists an assignment of n umpire crews (umpires, for short) to the games of T that satisfies all of the following conditions:

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$U_{8,0}$

Rounds						
0	1	2	3	4	5	6
(0, 7)	(1, 7)	(2, 7)	(3, 7)	(4, 7)	(5, 7)	(6, 7)
(1, 6)	(2, 0)	(3, 1)	(4, 2)	(5, 3)	(6, 4)	(0, 5)
(2, 5)	(3, 6)	(4, 0)	(5, 1)	(6, 2)	(0, 3)	(1, 4)
(3, 4)	(4, 5)	(5, 6)	(6, 0)	(0, 1)	(1, 2)	(2, 3)

$U_{8,8}$

Rounds						
0	1	2	3	4	5	6
(8, 15)	(9, 15)	(10, 15)	(11, 15)	(12, 15)	(13, 15)	(14, 15)
(9, 14)	(10, 8)	(11, 9)	(12, 10)	(13, 11)	(14, 12)	(8, 13)
(10, 13)	(11, 14)	(12, 8)	(13, 9)	(14, 10)	(8, 11)	(9, 12)
(11, 12)	(12, 13)	(13, 14)	(14, 8)	(8, 9)	(9, 10)	(10, 11)

$U_{8,16}$

Rounds						
0	1	2	3	4	5	6
(16, 23)	(17, 23)	(18, 23)	(19, 23)	(20, 23)	(21, 23)	(22, 23)
(17, 22)	(18, 16)	(19, 17)	(20, 18)	(21, 19)	(22, 20)	(16, 21)
(18, 21)	(19, 22)	(20, 16)	(21, 17)	(22, 18)	(16, 19)	(17, 20)
(19, 20)	(20, 21)	(21, 22)	(22, 16)	(16, 17)	(17, 18)	(18, 19)

$U_{8,24}$

Rounds						
0	1	2	3	4	5	6
(24, 31)	(25, 31)	(26, 31)	(27, 31)	(28, 31)	(29, 31)	(30, 31)
(25, 30)	(26, 24)	(27, 25)	(28, 26)	(29, 27)	(30, 28)	(24, 29)
(26, 29)	(27, 30)	(28, 24)	(29, 25)	(30, 26)	(24, 27)	(25, 28)
(27, 28)	(28, 29)	(29, 30)	(30, 24)	(24, 25)	(25, 26)	(26, 27)

Fig. 1. Tournaments $U_{8,0}$, $U_{8,8}$, $U_{8,16}$, and $U_{8,24}$.

(i) In every round of T, each umpire is assigned to exactly one game, and each game is assigned to exactly one umpire;
(ii) Each umpire visits the home venue of every team at least once;
(iii) No umpire visits a venue more than once in any sequence of $n - d_1$ consecutive rounds;
(iv) No umpire sees a team more than once in any sequence of $\lfloor n/2 \rfloor - d_2$ consecutive rounds;
(v) The total distance traveled by the n umpires during T is less than or equal to ℓ.

Our main contribution is to prove that the decision version of the TUP is an $\mathcal{NP}$-complete problem when $d_1 \le n/2$ and $d_2 = \lfloor n/2 \rfloor - 1$.

The remainder of this paper is organized as follows. Section 2 introduces some notation used throughout the paper and establishes a few preliminary results. Section 3 presents an $\mathcal{NP}$-complete problem that can be reduced to the TUP, followed by the proof of our main result. Finally, we conclude the paper and propose future research directions in Section 4.

2. Notation and preliminary results

In this section we introduce some notation that will be used in our main result and prove a number of auxiliary results.

Let T be a tournament with $2n$ teams and m rounds. Then, T can be defined as a sequence of sets of ordered pairs by writing $T = S_0, S_1, \ldots, S_{m-1}$, where S_s contains the games that take place in the $(s+1)$-th round.[1] We assume that the first team in each ordered pair is the home team. Let $C = \{(i_0, j_0), (i_1, j_1), \ldots, (i_{v-1}, j_{v-1})\}$ be a set with v ordered pairs. We denote by $\bar{C}$ the set obtained from C by reversing the order of the elements in each ordered pair in C. Therefore, $\bar{C} = \{(j_0, i_0), (j_1, i_1), \ldots, (j_{v-1}, i_{v-1})\}$. Using this notation, the reversal of the home venues of T can be denoted by $\bar{T} = \bar{S}_0, \bar{S}_1, \ldots, \bar{S}_{m-1}$. In other words, for every pair of teams i and j, if i plays at home against j in round s of T, then j plays at home against i in round s of $\bar{T}$.

A single (double) round-robin tournament is a tournament in which each team plays against each other team exactly once (twice: once at each team's home venue). Eqs. (1)–(3) define a constructive way of creating a single round-robin tournament $U_{a,b}$ with an even number of teams $a \ge 2$, $a-1$ rounds, and team IDs ranging from b to $b+a-1$:

$$U_{a,b} = U_{a,b}[0, a-2], \tag{1}$$

$$U_{a,b}[s_1, s_2] = Q_{a,b}[s_1], Q_{a,b}[s_1+1], \ldots, Q_{a,b}[s_2], \quad \forall 0 \le s_1 \le s_2 \le a-2, \tag{2}$$

$$\begin{aligned} Q_{a,b}[s] = \Big\{ & \big(b + (s \bmod (a-1)), b+a-1\big), \\ & \big(b + ((s+1) \bmod (a-1)), b + ((s+a-2) \bmod (a-1))\big), \\ & \big(b + ((s+2) \bmod (a-1)), b + ((s+a-3) \bmod (a-1))\big), \\ & \vdots \\ & \big(b + ((s+a/2-1) \bmod (a-1)), b + ((s+a-a/2) \bmod (a-1))\big)\Big\}, \quad \forall 0 \le s \le a-2. \end{aligned} \tag{3}$$

This algebraic definition results in a method equivalent to the well-known circle/polygon method, also known in literature as Kirkman's method, which was first introduced in [8]. Fig. 1 illustrates four 8-team $U_{a,b}$ tournaments, and Lemma 1 asserts the correctness of (1)–(3).

[1] Although we use round indices starting at zero, we avoid referring to round 0 as the zero-th round. Hence, S_0 is the first round, S_1 is the second round, and so on. The same applies to other ordinal indices throughout the paper.

Lemma 1. *Given an even number of teams $a \geq 2$ with IDs represented by consecutive numbers starting at $b \geq 0$, $U_{a,b}$ is a single round-robin tournament among those teams.*

Proof. See [9]. □

In addition to U, we now define another kind of tournament denoted by P. Later on, we will combine U and P tournaments of different sizes to create a large double round-robin tournament that will form the basis of our main proof.

Eqs. (4)–(6) define a tournament $P_{a,b}$ with $a > 0$ rounds and $2a$ teams with consecutive IDs starting at $b \geq 0$. The tournament is such that each one of the first a teams, namely $b, \ldots, b+a-1$, plays against each one of the next a teams, namely $b+a, \ldots, b+2a-1$, exactly once. Moreover, teams $b, \ldots, b+a-1$ do not play against each other, and neither do teams $b+a, \ldots, b+2a-1$.

$$P_{a,b} = P_{a,b}[0, a-1], \tag{4}$$

$$P_{a,b}[s_1, s_2] = X_{a,b}[s_1], X_{a,b}[s_1+1], \ldots, X_{a,b}[s_2], \quad \forall 0 \leq s_1 \leq s_2 \leq a-1, \tag{5}$$

$$\begin{aligned} X_{a,b}[s] = \big\{&\big(b+0, b+a+(s \bmod a)\big), \\ &\big(b+1, b+a+\big((s+1) \bmod a\big)\big), \\ &\vdots \\ &\big(b+a-1, b+a+\big((s+a-1) \bmod a\big)\big)\big\}, \quad \forall 0 \leq s \leq a-1. \end{aligned} \tag{6}$$

Note that every team plays exactly once in each round of a $P_{a,b}$ tournament. Fig. 2 presents three examples of $P_{a,b}$ tournaments, two with eight teams and one with sixteen teams. The notation $P_{a,b}[s_1, s_2]$ from (5) is used to represent the stretch of a $P_{a,b}$ tournament that extends from round s_1 to round s_2.

We are now ready to explain how U and P tournaments can be combined to produce a double round-robin tournament T. We begin by defining three operations that apply to generic sequences of elements. Let $A = A_1, A_2, \ldots, A_g$ and $B = B_1, B_2, \ldots, B_h$ be two sequences with g and h elements each. The *concatenation* operation $A \oplus B$ produces $A_1, A_2, \ldots, A_g, B_1, B_2, \ldots, B_h$. The *interleaving* operation $A \oslash B$ produces $A_1, B_1, A_2, B_2, \ldots, A_g, B_h$ when $g = h$, and produces $A_1, B_1, A_2, B_2, \ldots, A_h, B_h, A_{h+1}, A_{h+2}, \ldots, A_g$ when $g > h$. (The case $g < h$ works analogously.) Finally, when A and B are sequences of sets, the *pairwise union* operation $A \odot B$ produces $(A_1 \cup B_1), (A_2 \cup B_2), \ldots, (A_g \cup B_h)$ when $g = h$.

We combine tournaments $U_{k,0}$, $\overline{U}_{k,0}$, $U_{k,k}$, $\overline{U}_{k,k}$, $U_{k,2k}$, $\overline{U}_{k,2k}$, $U_{k,3k}$, $\overline{U}_{k,3k}$, $P_{k,0}$, $\overline{P}_{k,0}$, $P_{k,2k}$, $\overline{P}_{k,2k}$, $P_{2k,0}$, and $\overline{P}_{2k,0}$ to obtain a tournament T with $4k$ teams. Figs. 1 and 2 illustrate tournaments $U_{k,0}$, $U_{k,k}$, $U_{k,2k}$, $U_{k,3k}$, $P_{k,0}$, $P_{k,2k}$, and $P_{2k,0}$ for $k = 8$. We omit illustrations of $\overline{U}_{k,0}$, $\overline{U}_{k,k}$, $\overline{U}_{k,2k}$, $\overline{U}_{k,3k}$, $\overline{P}_{k,0}$, $\overline{P}_{k,2k}$, and $\overline{P}_{2k,0}$ because they are equal to the previous tournaments with home venues reversed. Tournament T is defined by (7)–(10). Fig. 3 shows all games of T for $k = 8$, while also indicating which term from (7)–(10) created the games underneath each stretch of rounds.

$$T_1 = \overline{P}_{2k,0}[0, k-1] \oslash (\overline{U}_{k,0} \odot \overline{U}_{k,k} \odot \overline{U}_{k,2k} \odot \overline{U}_{k,3k}), \tag{7}$$

$$T_2 = \overline{P}_{2k,0}[k, 2k-1] \oslash (U_{k,0} \odot U_{k,k} \odot U_{k,2k} \odot U_{k,3k}), \tag{8}$$

$$T_3 = (\overline{P}_{k,0} \odot \overline{P}_{k,2k}) \oslash (P_{k,0} \odot P_{k,2k}), \tag{9}$$

$$T = T_1 \oplus P_{2k,0}[0, k-1] \oplus T_2 \oplus P_{2k,0}[k, 2k-1] \oplus T_3. \tag{10}$$

Theorem 1. *T is a double round-robin tournament.*

Proof. By definition and Lemma 1, $U_{k,0}$ and $U_{k,k}$ are single round-robin tournaments with different teams. $U_{k,0} \odot U_{k,k}$ is a tournament with $k-1$ rounds in which each pair of teams in the interval $[0, k-1]$ plays exactly once, as does each pair of teams in the interval $[k, 2k-1]$. For this tournament to become a round-robin tournament we need the teams in $[0, k-1]$ to play the teams in $[k, 2k-1]$ exactly once during an additional k rounds. But this is exactly what happens during the rounds defined by $P_{k,0}$. Therefore, $(U_{k,0} \odot U_{k,k}) \oplus P_{k,0}$ is a single round-robin tournament. Tournament $(U_{k,2k} \odot U_{k,3k}) \oplus P_{k,2k}$ corresponds to tournament $(U_{k,0} \odot U_{k,k}) \oplus P_{k,0}$ with all team IDs increased by $2k$. This means that $(U_{k,2k} \odot U_{k,3k}) \oplus P_{k,2k}$ is also a single round-robin tournament, and all of its teams are different from the teams in $(U_{k,0} \odot U_{k,k}) \oplus P_{k,0}$. We combine these two tournaments in a manner that is similar to the way $U_{k,0}$ and $U_{k,k}$ were combined to obtain a larger single round-robin tournament. This way, the result of $(((U_{k,0} \odot U_{k,k}) \oplus P_{k,0}) \odot ((U_{k,2k} \odot U_{k,3k}) \oplus P_{k,2k})) \oplus P_{2k,0}$ is a single round-robin tournament as well. Note that $(((U_{k,0} \odot U_{k,k}) \oplus P_{k,0}) \odot ((U_{k,2k} \odot U_{k,3k}) \oplus P_{k,2k})) \oplus P_{2k,0} = (U_{k,0} \odot U_{k,k} \odot U_{k,2k} \odot U_{k,3k}) \oplus (P_{k,0} \odot P_{k,2k}) \oplus P_{2k,0}$. To obtain a double round-robin tournament, we combine the latter tournament with a copy of itself that has the home venues reversed. In the resulting tournament, each team plays against every other team, once at home and once on the road. The result is $(U_{k,0} \odot U_{k,k} \odot U_{k,2k} \odot U_{k,3k}) \oplus (P_{k,0} \odot P_{k,2k}) \oplus P_{2k,0} \oplus (\overline{U}_{k,0} \odot \overline{U}_{k,k} \odot \overline{U}_{k,2k} \odot \overline{U}_{k,3k}) \oplus (\overline{P}_{k,0} \odot \overline{P}_{k,2k}) \oplus \overline{P}_{2k,0}$. Finally, note that we can obtain T by simply rearranging the order of the rounds in this last tournament, which completes the proof. □

$P_{8,0}$

Rounds							
0	1	2	3	4	5	6	7
(0, 8)	(0, 9)	(0, 10)	(0, 11)	(0, 12)	(0, 13)	(0, 14)	(0, 15)
(1, 9)	(1, 10)	(1, 11)	(1, 12)	(1, 13)	(1, 14)	(1, 15)	(1, 8)
(2, 10)	(2, 11)	(2, 12)	(2, 13)	(2, 14)	(2, 15)	(2, 8)	(2, 9)
(3, 11)	(3, 12)	(3, 13)	(3, 14)	(3, 15)	(3, 8)	(3, 9)	(3, 10)
(4, 12)	(4, 13)	(4, 14)	(4, 15)	(4, 8)	(4, 9)	(4, 10)	(4, 11)
(5, 13)	(5, 14)	(5, 15)	(5, 8)	(5, 9)	(5, 10)	(5, 11)	(5, 12)
(6, 14)	(6, 15)	(6, 8)	(6, 9)	(6, 10)	(6, 11)	(6, 12)	(6, 13)
(7, 15)	(7, 8)	(7, 9)	(7, 10)	(7, 11)	(7, 12)	(7, 13)	(7, 14)

$P_{8,16}$

Rounds							
0	1	2	3	4	5	6	7
(16, 24)	(16, 25)	(16, 26)	(16, 27)	(16, 28)	(16, 29)	(16, 30)	(16, 31)
(17, 25)	(17, 26)	(17, 27)	(17, 28)	(17, 29)	(17, 30)	(17, 31)	(17, 24)
(18, 26)	(18, 27)	(18, 28)	(18, 29)	(18, 30)	(18, 31)	(18, 24)	(18, 25)
(19, 27)	(19, 28)	(19, 29)	(19, 30)	(19, 31)	(19, 24)	(19, 25)	(19, 26)
(20, 28)	(20, 29)	(20, 30)	(20, 31)	(20, 24)	(20, 25)	(20, 26)	(20, 27)
(21, 29)	(21, 30)	(21, 31)	(21, 24)	(21, 25)	(21, 26)	(21, 27)	(21, 28)
(22, 30)	(22, 31)	(22, 24)	(22, 25)	(22, 26)	(22, 27)	(22, 28)	(22, 29)
(23, 31)	(23, 24)	(23, 25)	(23, 26)	(23, 27)	(23, 28)	(23, 29)	(23, 30)

$P_{16,0}$

Rounds															
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
(0, 16)	(0, 17)	(0, 18)	(0, 19)	(0, 20)	(0, 21)	(0, 22)	(0, 23)	(0, 24)	(0, 25)	(0, 26)	(0, 27)	(0, 28)	(0, 29)	(0, 30)	(0, 31)
(1, 17)	(1, 18)	(1, 19)	(1, 20)	(1, 21)	(1, 22)	(1, 23)	(1, 24)	(1, 25)	(1, 26)	(1, 27)	(1, 28)	(1, 29)	(1, 30)	(1, 31)	(1, 16)
(2, 18)	(2, 19)	(2, 20)	(2, 21)	(2, 22)	(2, 23)	(2, 24)	(2, 25)	(2, 26)	(2, 27)	(2, 28)	(2, 29)	(2, 30)	(2, 31)	(2, 16)	(2, 17)
(3, 19)	(3, 20)	(3, 21)	(3, 22)	(3, 23)	(3, 24)	(3, 25)	(3, 26)	(3, 27)	(3, 28)	(3, 29)	(3, 30)	(3, 31)	(3, 16)	(3, 17)	(3, 18)
(4, 20)	(4, 21)	(4, 22)	(4, 23)	(4, 24)	(4, 25)	(4, 26)	(4, 27)	(4, 28)	(4, 29)	(4, 30)	(4, 31)	(4, 16)	(4, 17)	(4, 18)	(4, 19)
(5, 21)	(5, 22)	(5, 23)	(5, 24)	(5, 25)	(5, 26)	(5, 27)	(5, 28)	(5, 29)	(5, 30)	(5, 31)	(5, 16)	(5, 17)	(5, 18)	(5, 19)	(5, 20)
(6, 22)	(6, 23)	(6, 24)	(6, 25)	(6, 26)	(6, 27)	(6, 28)	(6, 29)	(6, 30)	(6, 31)	(6, 16)	(6, 17)	(6, 18)	(6, 19)	(6, 20)	(6, 21)
(7, 23)	(7, 24)	(7, 25)	(7, 26)	(7, 27)	(7, 28)	(7, 29)	(7, 30)	(7, 31)	(7, 16)	(7, 17)	(7, 18)	(7, 19)	(7, 20)	(7, 21)	(7, 22)
(8, 24)	(8, 25)	(8, 26)	(8, 27)	(8, 28)	(8, 29)	(8, 30)	(8, 31)	(8, 16)	(8, 17)	(8, 18)	(8, 19)	(8, 20)	(8, 21)	(8, 22)	(8, 23)
(9, 25)	(9, 26)	(9, 27)	(9, 28)	(9, 29)	(9, 30)	(9, 31)	(9, 16)	(9, 17)	(9, 18)	(9, 19)	(9, 20)	(9, 21)	(9, 22)	(9, 23)	(9, 24)
(10, 26)	(10, 27)	(10, 28)	(10, 29)	(10, 30)	(10, 31)	(10, 16)	(10, 17)	(10, 18)	(10, 19)	(10, 20)	(10, 21)	(10, 22)	(10, 23)	(10, 24)	(10, 25)
(11, 27)	(11, 28)	(11, 29)	(11, 30)	(11, 31)	(11, 16)	(11, 17)	(11, 18)	(11, 19)	(11, 20)	(11, 21)	(11, 22)	(11, 23)	(11, 24)	(11, 25)	(11, 26)
(12, 28)	(12, 29)	(12, 30)	(12, 31)	(12, 16)	(12, 17)	(12, 18)	(12, 19)	(12, 20)	(12, 21)	(12, 22)	(12, 23)	(12, 24)	(12, 25)	(12, 26)	(12, 27)
(13, 29)	(13, 30)	(13, 31)	(13, 16)	(13, 17)	(13, 18)	(13, 19)	(13, 20)	(13, 21)	(13, 22)	(13, 23)	(13, 24)	(13, 25)	(13, 26)	(13, 27)	(13, 28)
(14, 30)	(14, 31)	(14, 16)	(14, 17)	(14, 18)	(14, 19)	(14, 20)	(14, 21)	(14, 22)	(14, 23)	(14, 24)	(14, 25)	(14, 26)	(14, 27)	(14, 28)	(14, 29)
(15, 31)	(15, 16)	(15, 17)	(15, 18)	(15, 19)	(15, 20)	(15, 21)	(15, 22)	(15, 23)	(15, 24)	(15, 25)	(15, 26)	(15, 27)	(15, 28)	(15, 29)	(15, 30)

Fig. 2. Tournaments $P_{8,0}$, $P_{8,16}$, and $P_{16,0}$.

Rounds																						
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
$\overline{P}_{16,0}[0,7] \oslash (\overline{U}_{8,0} \odot \overline{U}_{8,8} \odot \overline{U}_{8,16} \odot \overline{U}_{8,24})$															$P_{16,0}[0,7]$							
(16, 0)	(7, 0)	(17, 0)	(7, 1)	(18, 0)	(7, 2)	(19, 0)	(7, 3)	(20, 0)	(7, 4)	(21, 0)	(7, 5)	(22, 0)	(7, 6)	(23, 0)	(0, 16)	(0, 17)	(0, 18)	(0, 19)	(0, 20)	(0, 21)	(0, 22)	(0, 23)
(17, 1)	(6, 1)	(18, 1)	(0, 2)	(19, 1)	(1, 3)	(20, 1)	(2, 4)	(21, 1)	(3, 5)	(22, 1)	(4, 6)	(23, 1)	(5, 0)	(24, 1)	(1, 17)	(1, 18)	(1, 19)	(1, 20)	(1, 21)	(1, 22)	(1, 23)	(1, 24)
(18, 2)	(5, 2)	(19, 2)	(6, 3)	(20, 2)	(0, 4)	(21, 2)	(1, 5)	(22, 2)	(2, 6)	(23, 2)	(3, 0)	(24, 2)	(4, 1)	(25, 2)	(2, 18)	(2, 19)	(2, 20)	(2, 21)	(2, 22)	(2, 23)	(2, 24)	(2, 25)
(19, 3)	(4, 3)	(20, 3)	(5, 4)	(21, 3)	(6, 5)	(22, 3)	(0, 6)	(23, 3)	(1, 0)	(24, 3)	(2, 1)	(25, 3)	(3, 2)	(26, 3)	(3, 19)	(3, 20)	(3, 21)	(3, 22)	(3, 23)	(3, 24)	(3, 25)	(3, 26)
(20, 4)	(15, 8)	(21, 4)	(15, 9)	(22, 4)	(15, 10)	(23, 4)	(15, 11)	(24, 4)	(15, 12)	(25, 4)	(15, 13)	(26, 4)	(15, 14)	(27, 4)	(4, 20)	(4, 21)	(4, 22)	(4, 23)	(4, 24)	(4, 25)	(4, 26)	(4, 27)
(21, 5)	(14, 9)	(22, 5)	(8, 10)	(23, 5)	(9, 11)	(24, 5)	(10, 12)	(25, 5)	(11, 13)	(26, 5)	(12, 14)	(27, 5)	(13, 8)	(28, 5)	(5, 21)	(5, 22)	(5, 23)	(5, 24)	(5, 25)	(5, 26)	(5, 27)	(5, 28)
(22, 6)	(13, 10)	(23, 6)	(14, 11)	(24, 6)	(8, 12)	(25, 6)	(9, 13)	(26, 6)	(10, 14)	(27, 6)	(11, 8)	(28, 6)	(12, 9)	(29, 6)	(6, 22)	(6, 23)	(6, 24)	(6, 25)	(6, 26)	(6, 27)	(6, 28)	(6, 29)
(23, 7)	(12, 11)	(24, 7)	(13, 12)	(25, 7)	(14, 13)	(26, 7)	(8, 14)	(27, 7)	(9, 8)	(28, 7)	(10, 9)	(29, 7)	(11, 10)	(30, 7)	(7, 23)	(7, 24)	(7, 25)	(7, 26)	(7, 27)	(7, 28)	(7, 29)	(7, 30)
(24, 8)	(23, 16)	(25, 8)	(23, 17)	(26, 8)	(23, 18)	(27, 8)	(23, 19)	(28, 8)	(23, 20)	(29, 8)	(23, 21)	(30, 8)	(23, 22)	(31, 8)	(8, 24)	(8, 25)	(8, 26)	(8, 27)	(8, 28)	(8, 29)	(8, 30)	(8, 31)
(25, 9)	(22, 17)	(26, 9)	(16, 18)	(27, 9)	(17, 19)	(28, 9)	(18, 20)	(29, 9)	(19, 21)	(30, 9)	(20, 22)	(31, 9)	(21, 16)	(16, 9)	(9, 25)	(9, 26)	(9, 27)	(9, 28)	(9, 29)	(9, 30)	(9, 31)	(9, 16)
(26, 10)	(21, 18)	(27, 10)	(22, 19)	(28, 10)	(16, 20)	(29, 10)	(17, 21)	(30, 10)	(18, 22)	(31, 10)	(19, 16)	(16, 10)	(20, 17)	(17, 10)	(10, 26)	(10, 27)	(10, 28)	(10, 29)	(10, 30)	(10, 31)	(10, 16)	(10, 17)
(27, 11)	(20, 19)	(28, 11)	(21, 20)	(29, 11)	(22, 21)	(30, 11)	(16, 22)	(31, 11)	(17, 16)	(16, 11)	(18, 17)	(17, 11)	(19, 18)	(18, 11)	(11, 27)	(11, 28)	(11, 29)	(11, 30)	(11, 31)	(11, 16)	(11, 17)	(11, 18)
(28, 12)	(31, 24)	(29, 12)	(31, 25)	(30, 12)	(31, 26)	(31, 12)	(31, 27)	(16, 12)	(31, 28)	(17, 12)	(31, 29)	(18, 12)	(31, 30)	(19, 12)	(12, 28)	(12, 29)	(12, 30)	(12, 31)	(12, 16)	(12, 17)	(12, 18)	(12, 19)
(29, 13)	(30, 25)	(30, 13)	(24, 26)	(31, 13)	(25, 27)	(16, 13)	(26, 28)	(17, 13)	(27, 29)	(18, 13)	(28, 30)	(19, 13)	(29, 24)	(20, 13)	(13, 29)	(13, 30)	(13, 31)	(13, 16)	(13, 17)	(13, 18)	(13, 19)	(13, 20)
(30, 14)	(29, 26)	(31, 14)	(30, 27)	(16, 14)	(24, 28)	(17, 14)	(25, 29)	(18, 14)	(26, 30)	(19, 14)	(27, 24)	(20, 14)	(28, 25)	(21, 14)	(14, 30)	(14, 31)	(14, 16)	(14, 17)	(14, 18)	(14, 19)	(14, 20)	(14, 21)
(31, 15)	(28, 27)	(16, 15)	(29, 28)	(17, 15)	(30, 29)	(18, 15)	(24, 30)	(19, 15)	(25, 24)	(20, 15)	(26, 25)	(21, 15)	(27, 26)	(22, 15)	(15, 31)	(15, 16)	(15, 17)	(15, 18)	(15, 19)	(15, 20)	(15, 21)	(15, 22)

Rounds																						
23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
$\overline{P}_{16,0}[8,15] \oslash (U_{8,0} \odot U_{8,8} \odot U_{8,16} \odot U_{8,24})$															$P_{16,0}[8,15]$							
(24, 0)	(0, 7)	(25, 0)	(1, 7)	(26, 0)	(2, 7)	(27, 0)	(3, 7)	(28, 0)	(4, 7)	(29, 0)	(5, 7)	(30, 0)	(6, 7)	(31, 0)	(0, 24)	(0, 25)	(0, 26)	(0, 27)	(0, 28)	(0, 29)	(0, 30)	(0, 31)
(25, 1)	(1, 6)	(26, 1)	(2, 0)	(27, 1)	(3, 1)	(28, 1)	(4, 2)	(29, 1)	(5, 3)	(30, 1)	(6, 4)	(31, 1)	(0, 5)	(16, 1)	(1, 25)	(1, 26)	(1, 27)	(1, 28)	(1, 29)	(1, 30)	(1, 31)	(1, 16)
(26, 2)	(2, 5)	(27, 2)	(3, 6)	(28, 2)	(4, 0)	(29, 2)	(5, 1)	(30, 2)	(6, 2)	(31, 2)	(0, 3)	(16, 2)	(1, 4)	(17, 2)	(2, 26)	(2, 27)	(2, 28)	(2, 29)	(2, 30)	(2, 31)	(2, 16)	(2, 17)
(27, 3)	(3, 4)	(28, 3)	(4, 5)	(29, 3)	(5, 6)	(30, 3)	(6, 0)	(31, 3)	(0, 1)	(16, 3)	(1, 2)	(17, 3)	(2, 3)	(18, 3)	(3, 27)	(3, 28)	(3, 29)	(3, 30)	(3, 31)	(3, 16)	(3, 17)	(3, 18)
(28, 4)	(8, 15)	(29, 4)	(9, 15)	(30, 4)	(10, 15)	(31, 4)	(11, 15)	(16, 4)	(12, 15)	(17, 4)	(13, 15)	(18, 4)	(14, 15)	(19, 4)	(4, 28)	(4, 29)	(4, 30)	(4, 31)	(4, 16)	(4, 17)	(4, 18)	(4, 19)
(29, 5)	(9, 14)	(30, 5)	(10, 8)	(31, 5)	(11, 9)	(16, 5)	(12, 10)	(17, 5)	(13, 11)	(18, 5)	(14, 12)	(19, 5)	(8, 13)	(20, 5)	(5, 29)	(5, 30)	(5, 31)	(5, 16)	(5, 17)	(5, 18)	(5, 19)	(5, 20)
(30, 6)	(10, 13)	(31, 6)	(11, 14)	(16, 6)	(12, 8)	(17, 6)	(13, 9)	(18, 6)	(14, 10)	(19, 6)	(8, 11)	(20, 6)	(9, 12)	(21, 6)	(6, 30)	(6, 31)	(6, 16)	(6, 17)	(6, 18)	(6, 19)	(6, 20)	(6, 21)
(31, 7)	(11, 12)	(16, 7)	(12, 13)	(17, 7)	(13, 14)	(18, 7)	(14, 8)	(19, 7)	(8, 9)	(20, 7)	(9, 10)	(21, 7)	(10, 11)	(22, 7)	(7, 31)	(7, 16)	(7, 17)	(7, 18)	(7, 19)	(7, 20)	(7, 21)	(7, 22)
(16, 8)	(16, 23)	(17, 8)	(17, 23)	(18, 8)	(18, 23)	(19, 8)	(19, 23)	(20, 8)	(20, 23)	(21, 8)	(21, 23)	(22, 8)	(22, 23)	(23, 8)	(8, 16)	(8, 17)	(8, 18)	(8, 19)	(8, 20)	(8, 21)	(8, 22)	(8, 23)
(17, 9)	(17, 22)	(18, 9)	(18, 16)	(19, 9)	(19, 17)	(20, 9)	(20, 18)	(21, 9)	(21, 19)	(22, 9)	(22, 20)	(23, 9)	(16, 21)	(24, 9)	(9, 17)	(9, 18)	(9, 19)	(9, 20)	(9, 21)	(9, 22)	(9, 23)	(9, 24)
(18, 10)	(18, 21)	(19, 10)	(19, 22)	(20, 10)	(20, 16)	(21, 10)	(21, 17)	(22, 10)	(22, 18)	(23, 10)	(16, 19)	(24, 10)	(17, 20)	(25, 10)	(10, 18)	(10, 19)	(10, 20)	(10, 21)	(10, 22)	(10, 23)	(10, 24)	(10, 25)
(19, 11)	(19, 20)	(20, 11)	(20, 21)	(21, 11)	(21, 22)	(22, 11)	(22, 16)	(23, 11)	(16, 17)	(24, 11)	(17, 18)	(25, 11)	(18, 19)	(26, 11)	(11, 19)	(11, 20)	(11, 21)	(11, 22)	(11, 23)	(11, 24)	(11, 25)	(11, 26)
(20, 12)	(24, 31)	(21, 12)	(25, 31)	(22, 12)	(26, 31)	(23, 12)	(27, 31)	(24, 12)	(28, 31)	(25, 12)	(29, 31)	(26, 12)	(30, 31)	(27, 12)	(12, 20)	(12, 21)	(12, 22)	(12, 23)	(12, 24)	(12, 25)	(12, 26)	(12, 27)
(21, 13)	(25, 30)	(22, 13)	(26, 24)	(23, 13)	(27, 25)	(24, 13)	(28, 26)	(25, 13)	(29, 27)	(26, 13)	(30, 28)	(27, 13)	(24, 29)	(28, 13)	(13, 21)	(13, 22)	(13, 23)	(13, 24)	(13, 25)	(13, 26)	(13, 27)	(13, 28)
(22, 14)	(26, 29)	(23, 14)	(27, 30)	(24, 14)	(28, 24)	(25, 14)	(29, 25)	(26, 14)	(30, 26)	(27, 14)	(24, 27)	(28, 14)	(25, 28)	(29, 14)	(14, 22)	(14, 23)	(14, 24)	(14, 25)	(14, 26)	(14, 27)	(14, 28)	(14, 29)
(23, 15)	(27, 28)	(24, 15)	(28, 29)	(25, 15)	(29, 30)	(26, 15)	(30, 24)	(27, 15)	(24, 25)	(28, 15)	(25, 26)	(29, 15)	(26, 27)	(30, 15)	(15, 23)	(15, 24)	(15, 25)	(15, 26)	(15, 27)	(15, 28)	(15, 29)	(15, 30)

Rounds															
46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61
$(\overline{P}_{8,0} \odot \overline{P}_{8,16}) \oslash (P_{8,0} \odot P_{8,16})$															
(8, 0)	(0, 8)	(9, 0)	(0, 9)	(10, 0)	(0, 10)	(11, 0)	(0, 11)	(12, 0)	(0, 12)	(13, 0)	(0, 13)	(14, 0)	(0, 14)	(15, 0)	(0, 15)
(9, 1)	(1, 9)	(10, 1)	(1, 10)	(11, 1)	(1, 11)	(12, 1)	(1, 12)	(13, 1)	(1, 13)	(14, 1)	(1, 14)	(15, 1)	(1, 15)	(8, 1)	(1, 8)
(10, 2)	(2, 10)	(11, 2)	(2, 11)	(12, 2)	(2, 12)	(13, 2)	(2, 13)	(14, 2)	(2, 14)	(15, 2)	(2, 15)	(8, 2)	(2, 8)	(9, 2)	(2, 9)
(11, 3)	(3, 11)	(12, 3)	(3, 12)	(13, 3)	(3, 13)	(14, 3)	(3, 14)	(15, 3)	(3, 15)	(8, 3)	(3, 8)	(9, 3)	(3, 9)	(10, 3)	(3, 10)
(12, 4)	(4, 12)	(13, 4)	(4, 13)	(14, 4)	(4, 14)	(15, 4)	(4, 15)	(8, 4)	(4, 8)	(9, 4)	(4, 9)	(10, 4)	(4, 10)	(11, 4)	(4, 11)
(13, 5)	(5, 13)	(14, 5)	(5, 14)	(15, 5)	(5, 15)	(8, 5)	(5, 8)	(9, 5)	(5, 9)	(10, 5)	(5, 10)	(11, 5)	(5, 11)	(12, 5)	(5, 12)
(14, 6)	(6, 14)	(15, 6)	(6, 15)	(8, 6)	(6, 8)	(9, 6)	(6, 9)	(10, 6)	(6, 10)	(11, 6)	(6, 11)	(12, 6)	(6, 12)	(13, 6)	(6, 13)
(15, 7)	(7, 15)	(8, 7)	(7, 8)	(9, 7)	(7, 9)	(10, 7)	(7, 10)	(11, 7)	(7, 11)	(12, 7)	(7, 12)	(13, 7)	(7, 13)	(14, 7)	(7, 14)
(24, 16)	(16, 24)	(25, 16)	(16, 25)	(26, 16)	(16, 26)	(27, 16)	(16, 27)	(28, 16)	(16, 28)	(29, 16)	(16, 29)	(30, 16)	(16, 30)	(31, 16)	(16, 31)
(25, 17)	(17, 25)	(26, 17)	(17, 26)	(27, 17)	(17, 27)	(28, 17)	(17, 28)	(29, 17)	(17, 29)	(30, 17)	(17, 30)	(31, 17)	(17, 31)	(24, 17)	(17, 24)
(26, 18)	(18, 26)	(27, 18)	(18, 27)	(28, 18)	(18, 28)	(29, 18)	(18, 29)	(30, 18)	(18, 30)	(31, 18)	(18, 31)	(24, 18)	(18, 24)	(25, 18)	(18, 25)
(27, 19)	(19, 27)	(28, 19)	(19, 28)	(29, 19)	(19, 29)	(30, 19)	(19, 30)	(31, 19)	(19, 31)	(24, 19)	(19, 24)	(25, 19)	(19, 25)	(26, 19)	(19, 26)
(28, 20)	(20, 28)	(29, 20)	(20, 29)	(30, 20)	(20, 30)	(31, 20)	(20, 31)	(24, 20)	(20, 24)	(25, 20)	(20, 25)	(26, 20)	(20, 26)	(27, 20)	(20, 27)
(29, 21)	(21, 29)	(30, 21)	(21, 30)	(31, 21)	(21, 31)	(24, 21)	(21, 24)	(25, 21)	(21, 25)	(26, 21)	(21, 26)	(27, 21)	(21, 27)	(28, 21)	(21, 28)
(30, 22)	(22, 30)	(31, 22)	(22, 31)	(24, 22)	(22, 24)	(25, 22)	(22, 25)	(26, 22)	(22, 26)	(27, 22)	(22, 27)	(28, 22)	(22, 28)	(29, 22)	(22, 29)
(31, 23)	(23, 31)	(24, 23)	(23, 24)	(25, 23)	(23, 25)	(26, 23)	(23, 26)	(27, 23)	(23, 27)	(28, 23)	(23, 28)	(29, 23)	(23, 29)	(30, 23)	(23, 30)

Fig. 3. Tournament T with $k = 8$.

In Section 3 we use tournament T to create an instance of the TUP to which an instance of another $\mathcal{NP}$-complete problem can be reduced in polynomial time.

3. Polynomial-time reduction

To show that the TUP is $\mathcal{NP}$-complete, we show that the problem of determining whether or not there exists a hamiltonian circuit in a graph with an even number of vertices and at least one universal vertex (i.e. a vertex adjacent to all other vertices) is $\mathcal{NP}$-complete, and then reduce this latter problem to the TUP in polynomial time.

Lemma 2. *Deciding whether or not a graph with an even number of vertices and at least one universal vertex has a hamiltonian circuit is an $\mathcal{NP}$-complete problem.*

Proof. Deciding whether or not a general graph has a hamiltonian path is an $\mathcal{NP}$-complete problem (see [10]). Let $G = (V, E)$ be a general graph and, therefore, an instance of the hamiltonian path problem. Starting from G we will construct, in polynomial time, a graph G' that is an instance of the problem described in this lemma. We first set $G' = G$ and consider two cases:

Case 1: G has an odd number of vertices. Add a universal vertex p to G', that is, p is a new vertex adjacent to all other vertices in G'. This takes $O(|V|)$ time.

Case 2: G has an even number of vertices. Add four new vertices to G': p, q, r, and s. Make p and q universal vertices (adjacent to each other, as well as r and s), and add an edge between r and s in G'. This also takes $O(|V|)$ time.

($\Rightarrow$) If G has a hamiltonian path, this path is also in G'. In Case 1, by connecting both endpoints of this path to p we create a hamiltonian circuit in G'. In Case 2, we create a hamiltonian circuit in G' as follows: connect one of the endpoints of the hamiltonian path to p, connect p to s, s to r, r to q, and finally connect q to the other endpoint of the path.

($\Leftarrow$) Assume G' has a hamiltonian circuit. In Case 1, we simply remove p from the circuit to obtain a hamiltonian path in G. In Case 2, the only way to reach vertices r and s is via p and q. This means that any hamiltonian circuit in G' goes through p (or q), immediately followed by r and s (in any order), and then goes through q (or p). Hence, by removing p, q, r, and s from the hamiltonian circuit in G' we end up with a hamiltonian path in G. □

We now show how to convert, in polynomial time, an instance of the decision problem of Lemma 2 into an instance of the TUP with $d_1 \le n/2$ and $d_2 = \lfloor n/2 \rfloor - 1$.

Definition 2. Given a graph G with an even number of vertices $k \ge 4$ and at least one universal vertex, we define a TUP instance $I(G)$ as follows. Its tournament T is the $4k$-team tournament defined by (7)–(10) in which we create a one-to-one correspondence between teams $0, \dots, k-1$ and the vertices of G. The number of umpires is $n = 2k$, $0 \le d_1 \le n/2$, and $d_2 = \lfloor n/2 \rfloor - 1$. Finally, for each pair of teams i and j, $d_{ij} = 0$ if $0 \le i, j \le k-1$ and the vertices corresponding to teams i and j are adjacent in G. Otherwise, $d_{ij} = 1$.

Theorem 2. *Let G be a graph with an even number of vertices $k \ge 4$ and at least one universal vertex. G has a hamiltonian circuit if, and only if, the TUP instance $I(G)$ of Definition 2 has an optimal solution with total travel distance equal to $n(4n-3) - 2k(k-1)$, where $n = 2k$ is the number of umpires.*

Before proving Theorem 2 we need to define two auxiliary numerical sequences and establish some of their key properties.

Given a positive integer a and three non-negative integers b, c, and d, let $Z_{a,b,c,d}$ be the sequence defined as

$$Z_{a,b,c,d} = d + (b \bmod a), d + \big((b+1) \bmod a\big), \dots, d + \big((b+c-1) \bmod a\big). \tag{11}$$

Note that $Z_{a,b,c,d}$ has length c and its numbers belong to the interval $[d, d+a-1]$, with the starting number being affected by the value of b. The sequence consists of consecutive integers up until the point when its largest possible number is reached, after which the next number is the smallest in the sequence. Therefore, if $c \le a$, all numbers in $Z_{a,b,c,d}$ are different. Sequences defined by (11) have the following additional property.

Lemma 3. *Given two sequences $Z_{a,b,c,d}$ and $Z_{a+1,b,c+1,d}$, if $c \le a$ and $c + b \le 2a$, the i-th number in $Z_{a,b,c,d}$ occurs exactly once in $Z_{a+1,b,c+1,d}$, being either the i-th or the $(i+1)$-th number of the latter sequence.*

Proof. Consider sequences $Z_{a,0,2a,d} = d, d+1, \dots, d+a-1, d, d+1, \dots, d+a-1$ and $Z_{a+1,0,2a+1,d} = d, d+1, \dots, d+a-1, d+a, d, d+1, \dots, d+a-1$. The i-th number in $Z_{a,0,2a,d}$ is equal to the i-th or $(i+1)$-th number in $Z_{a+1,0,2a+1,d}$. Requiring that $c + b \le 2a$ implies that sequences $Z_{a,b,c,d}$ and $Z_{a+1,b,c+1,d}$ are, respectively, subsequences of $Z_{a,0,2a,d}$ and $Z_{a+1,0,2a+1,d}$ that start at the b-th number of these latter sequences. Therefore, the i-th number in $Z_{a,b,c,d}$ is equal to either the i-th or the $(i+1)$-th number in $Z_{a+1,b,c+1,d}$. In addition, because $c + 1 \le a + 1$, all numbers in $Z_{a+1,b,c+1,d}$ are different, implying that the i-th number in $Z_{a,b,c,d}$ occurs exactly once in $Z_{a+1,b,c+1,d}$. □

Our second auxiliary numerical sequence $Y_{a,b,c,d,e}$ is defined by (12)–(14), where $a \geq 4$ is an even positive integer, b, c, d, and e are non-negative integers, and $0 \leq b \leq a/2 - 1$.

$$Y_{a,b,c,d,e} = (c, d) \oplus Z_{a-1,a+b,a-2b-3,e}, \quad \text{if } b = 0, \tag{12}$$

$$Y_{a,b,c,d,e} = Z_{a-1,a-b-1,2b,e} \oplus (c, d) \oplus Z_{a-1,a+b,a-2b-3,e}, \quad \text{if } 1 \leq b \leq a/2 - 2, \tag{13}$$

$$Y_{a,b,c,d,e} = Z_{a-1,a-b-1,2b,e} \oplus (c), \quad \text{if } b = a/2 - 1. \tag{14}$$

The following two lemmas establish useful properties of $Y_{a,b,c,d,e}$ sequences.

Lemma 4. *Let $a \geq 4$ be an even integer, b, c, d, and e be non-negative integers, $c \neq d$, and $0 \leq b \leq a/2 - 1$. If c and d do not belong to the interval $[e, e + a - 2]$, the numbers in $Y_{a,b,c,d,e}$ are all different.*

Proof. Clearly, the permissible values for c and d do not appear in either $Z_{a-1,a-b-1,2b,e}$ or $Z_{a-1,a+b,a-2b-3,e}$. Note that $Z_{a-1,a-b-1,a-2,e}$, whose numbers are all different since it has $a - 2$ elements, is equivalent to $Z_{a-1,a-b-1,2b,e} \oplus Z_{a-1,a+b-1,1,e} \oplus Z_{a-1,a+b,a-2b-3,e}$. Because $Z_{a-1,a-b-1,2b,e}$ and $Z_{a-1,a+b,a-2b-3,e}$ are subsequences of $Z_{a-1,a-b-1,a-2,e}$, they do not contain repeated numbers; neither within themselves nor between themselves. □

Lemma 5. *Let $a \geq 4$ be an even integer, and b, c, d, and e be non-negative integers. Given a non-negative integer $i \leq a - 2$, the set consisting of the $(i + 1)$-th numbers in the $Y_{a,b,c,d,e}$ sequences obtained for each value of $b \in \{0, \ldots, a/2 - 1\}$ contains all of the numbers in the interval $[e + ((i + a/2) \bmod (a - 1)), e + ((i + a - 2) \bmod (a - 1))]$. This set also contains the number c when i is even, or the number d when i is odd. In addition, if neither c nor d belong to the interval $[e, e + a - 2]$, the $(i + 1)$-th numbers in all of these $Y_{a,b,c,d,e}$ sequences are different.*

Proof. By definition, the $(i + 1)$-th number in $Y_{a,\lfloor i/2 \rfloor,c,d,e}$ is c when i is even, or d when i is odd. The $(i + 1)$-th number in each of the $a/2 - 1$ remaining $Y_{a,b,c,d,e}$ sequences (for $b \neq \lfloor i/2 \rfloor$) can be determined as follows. If $i \leq 1$, the $(i + 1)$-th number in $Y_{a,b,c,d,e}$ for $1 \leq b \leq a/2 - 1$ is given by $e + (((a - b - 1) + i) \bmod (a - 1))$ because it is the $(i + 1)$-th number in the $Z_{a-1,a-b-1,2b,e}$ subsequence from (13) or (14). If $i = a - 2$, the $(i + 1)$-th number in $Y_{a,b,c,d,e}$ for $0 \leq b \leq a/2 - 2$ is given by $e + (((a + b) + i - (2b + 2)) \bmod (a - 1))$ because it is the $(i - (2b + 2) + 1)$-th number in the $Z_{a-1,a+b,a-2b-3,e}$ subsequence from (12) or (13). If $2 \leq i \leq a - 3$, both of the previous cases can happen because the $(i + 1)$-th number in $Y_{a,b,c,d,e}$, depending on the value of b, can be in the $Z_{a-1,a+b,a-2b-3,e}$ subsequence from (12) or (13), or in the $Z_{a-1,a-b-1,2b,e}$ subsequence from (13) or (14). The $(i + 1)$-th number in $Y_{a,b,c,d,e}$ for $0 \leq b \leq \lfloor i/2 \rfloor - 1$ is given by $e + (((a + b) + i - (2b + 2)) \bmod (a - 1))$, whereas it is given by $e + (((a - b - 1) + i) \bmod (a - 1))$ for $\lfloor i/2 \rfloor + 1 \leq b \leq a/2 - 1$. As we vary the value of $b \neq \lfloor i/2 \rfloor$, in decreasing order, for a fixed value of i as discussed above, we verify that the $(i + 1)$-th numbers obtained are all of the numbers in the interval $[e + ((i + a/2) \bmod (a - 1)), e + ((i + a - 2) \bmod (a - 1))]$. Finally, note that this interval contains $a/2 - 1$ distinct numbers which, from the statement of the lemma, include neither c nor d. Therefore, the $(i + 1)$-th numbers from the $a/2$ sequences $Y_{a,b,c,d,e}$ are all different. □

We are now ready to prove Theorem 2.

Proof of Theorem 2. Because tournament T has $4n - 2$ rounds, any solution to $I(G)$ has a total of $n(4n - 3)$ trips (n umpires, $(4n - 3)$ trips each). From the definition of d_{ij}, trip distances are equal to either 0 or 1. For simplicity, we refer to them as 0-trips and 1-trips, respectively. Given the way the rounds of T are organized, 0-trips only occur between consecutive rounds in the intervals $[2k - 1, 3k - 2]$ and $[5k - 2, 6k - 3]$ because these are the only rounds that include games at the home venues of teams $0, \ldots, k - 1$. Because at most k 0-trips take place between consecutive rounds in each of those two intervals, there can be at most $2k(k - 1)$ 0-trips during the entire tournament. Therefore, any solution with a total distance of $n(4n - 3) - 2k(k - 1)$ is optimal.

($\Leftarrow$) By hypothesis, there exists a solution S to $I(G)$ with total distance equal to $n(4n - 3) - 2k(k - 1)$. This implies that k umpires in S go on 0-trips between each pair of consecutive rounds in the intervals $[2k - 1, 3k - 2]$ and $[5k - 2, 6k - 3]$. Let u be the umpire who, in round $2k - 1$, is at the home venue of a team associated with a universal vertex of G. If u does not go on a 0-trip from round $2k - 1$ to round $2k$, there cannot be k 0-trips between these rounds, since u is at one of the k venues where 0-trips can originate. As a consequence, in round $2k$, u will again be at the home venue of a team in the interval $[0, k - 1]$. The previous argument can be re-applied as u travels from round $2k$ to round $2k + 1$, and all the way to round $3k - 2$: all of u's trips are 0-trips from/to home venues of teams numbered between 0 and $k - 1$. Since $d_1 \leq n/2$, we have $n - d_1 \geq n/2 = k$. Because S is a feasible solution to the TUP, constraint (iii) implies that the k venues visited by u from round $2k - 1$ to round $3k - 2$ are all different. Therefore, the route traveled by u corresponds to a hamiltonian path in G. Moreover, because u's starting venue in round $2k - 1$ corresponds to a universal vertex, the endpoints of this hamiltonian path can be connected to form a hamiltonian circuit in G.

($\Rightarrow$) By hypothesis, G has a hamiltonian circuit $C = v_0, v_1, \ldots, v_{k-1}, v_0$, where each v_i ($i = 0, \ldots, k - 1$) is the ID of the team corresponding to that particular vertex in the circuit. Starting from C, we will create a solution S to the TUP instance $I(G)$ with total travel distance equal to $n(4n - 3) - 2k(k - 1)$. (Recall that the $(i + 1)$-th venue visited by an umpire is reached in round i of T because round numbers start at zero.) For each umpire $u \in \{0, \ldots, 2k-1\}$ the sequence of home venues visited by u in S is given by

$$(Z_{k,u,k,3k} \oslash Y_{k,u,k-1,3k-1,0}) \oplus Z_{k,u,k,k} \oplus (Z_{k,u,k,3k} \oslash Z_{k-1,u,k-1,2k}) \oplus W_u \oplus (Z_{k,u+(k/2),k,k} \oslash Z_{k,u,k,2k}),$$
$$\forall 0 \le u \le k/2-1, \quad (15)$$

$$(Z_{k,u',k,2k} \oslash Y_{k,u',2k-1,4k-1,k}) \oplus W_{u'} \oplus (Z_{k,u',k,2k} \oslash Z_{k-1,u',k-1,3k}) \oplus Z_{k,u',k,k} \oplus (Z_{k,u',k,3k} \oslash Z_{k,u,k,0}),$$
$$\forall k/2 \le u \le k-1, \quad (16)$$

$$(Z_{k,u',k,3k} \oslash Y_{k,u-k,3k-1,k-1,2k}) \oplus Z_{k,u',k,k} \oplus (Z_{k,2k-u-1,k,2k} \oslash Z_{k-1,u-k,k-1,0}) \oplus Z_{k,u',k,k} \oplus (Z_{k,u',k,3k} \oslash Z_{k,u-k,k,0}),$$
$$\forall k \le u \le 3k/2-1, \quad (17)$$

$$(Z_{k,u-k,k,2k} \oslash Y_{k,u'',4k-1,2k-1,3k}) \oplus W_{u-k} \oplus (Z_{k,k-u''-1,k,3k} \oslash Z_{k-1,u'',k-1,k}) \oplus W_{u-k} \oplus (Z_{k,u'',k,k} \oslash Z_{k,u-k,k,2k}),$$
$$\forall 3k/2 \le u \le 2k-1, \quad (18)$$

where $u' = u - k/2$, $u'' = u - 3k/2$, and $W_a = v_{(a \bmod k)}, v_{((a+1) \bmod k)}, \ldots, v_{((a+k-1) \bmod k)}$ for $0 \le a \le k-1$. Fig. 4 illustrates S when $k = 8$ and $C = 0, 1, 2, 3, 4, 5, 6, 7, 0$, with each relevant subsequence from (15)–(18) specified above the venues visited by the corresponding umpires.

We now show that S is a feasible solution to instance $I(G)$ with total travel distance equal to $n(4n-3) - 2k(k-1)$. We do not need to worry about constraint (iv) because $d_2 = \lfloor n/2 \rfloor - 1$ implies that it is trivially satisfied.

Constraint (i) is satisfied by S if, for each round of T, the home venue of each game is assigned to a different umpire by (15)–(18). This requires that we match the home venues from the definition of T in (7)–(10) with the home venues from (15)–(18). To simplify this task, we create Table 1 in which the rounds of T are conveniently separated into groups. For each group, it shows the sub-tournaments in (7)–(10) that determine the venues where the games take place. Table 1 also presents the umpires assigned to each home venue, as well as the subsequences in (15)–(18) that define these assignments. In the ensuing discussion, we refer to Table 1 to demonstrate, based on the definitions of U, $\overline{U}$, P, $\overline{P}$, W, Y, and Z, that the assignments of umpires to home venues made by S agree with the games of T.

The home venues in the $(i+1)$-th rounds of $\overline{U}_{k,b}$ are $b + ((i + k/2) \bmod (k-1)), \ldots, b + ((i + k - 2) \bmod (k-1))$ and $b + k - 1$, for $0 \le i \le k-2$. Lemma 5 guarantees that, for $0 \le b \le k/2 - 1$, the $(i+1)$-th home venues in $Y_{k,b,c,d,e}$ are $e + ((i + k/2) \bmod (k-1)), \ldots, e + ((i + k - 2) \bmod (k-1))$ and c, if i is even, or d otherwise. Based on these two observations we can see that, during rounds $1, 3, \ldots, 2k-3$, the home venues in $U_{k,0}$ and $U_{k,2k}$ are correctly assigned by $Y_{k,u,k-1,3k-1,0}$ in (15) and by $Y_{k,u-k,3k-1,k-1,2k}$ in (17). Likewise, the home venues in $U_{k,k}$ and $U_{k,3k}$ are also properly assigned by $Y_{k,u',2k-1,4k-1,k}$ in (16) and by $Y_{k,u'',4k-1,2k-1,3k}$ in (18).

Analogously, the home venues in the $(i+1)$-th rounds of $U_{k,b}$ are $b + (i \bmod (k-1)), \ldots, b + ((i + k/2 - 1) \bmod (k-1))$, for $0 \le i \le k-2$, whereas the $(i+1)$-th venues in $Z_{k-1,b,k-1,d}$ are $d + (i \bmod (k-1)), \ldots, d + ((i + k/2 - 1) \bmod (k-1))$, for $0 \le b \le k/2 - 1$. Therefore, during rounds $3k, 3k+2, \ldots, 5k-4$, the home venues in $U_{k,0}$, $U_{k,k}$, $U_{k,2k}$, and $U_{k,3k}$ are correctly assigned by $Z_{k-1,u-k,k-1,0}$, $Z_{k-1,u'',k-1,k}$, $Z_{k-1,u,k-1,2k}$, and $Z_{k-1,u',k-1,3k}$, respectively. These latter sequences in turn being defined by (17), (18), (15), and (16), respectively.

To complete this part of the proof, we only need to verify the venue assignments for rounds defined by P and $\overline{P}$. The home venues of any round in $P_{a,b}$ and $\overline{P}_{a,b}$ are, respectively, $b, \ldots, b + a - 1$ and $b + a, \ldots, b + 2a - 1$. The $(i+1)$-th home venues in W_a, for $0 \le a, i \le k-1$, are $0, \ldots, k-1$, whereas the $(i+1)$-th home venues in $Z_{k,b,k,d}$, for $0 \le b \le k-1$, are $d + (i \bmod k), \ldots, d + ((i + k - 1) \bmod k)$, which simplify to $d, \ldots, d + k - 1$. These observations allow us to conclude that the venue assignments for the above rounds are correctly made by the W and Z sequences in Table 1. For example, during rounds $0, 2, \ldots, 2k-2$, whose games are defined by $\overline{P}_{2k,0}[0, k-1]$, the home venues $2k, \ldots, 3k-1$ are assigned by $Z_{k,u',k,2k}$ in (16), and by $Z_{k,u-k,k,2k}$ in (18), whereas venues $3k, \ldots, 4k-1$ are assigned by $Z_{k,u,k,3k}$ in (15), and by $Z_{k,u',k,3k}$ in (17). The assignments in the remaining rounds can be verified in a similar manner, which completes the argument that S satisfies constraint (i).

Sequences $Z_{k,u,k,3k}$, $Z_{k,u,k,k}$, W_u, and $Z_{k,u,k,2k}$ in (15), $Z_{k,u',k,2k}$, $W_{u'}$, $Z_{k,u',k,k}$, and $Z_{k,u',k,3k}$ in (16), $Z_{k,u',k,3k}$, $Z_{k,u',k,k}$, $Z_{k,2k-u-1,k,2k}$, and $Z_{k,u-k,k,0}$ in (17), $Z_{k,u-k,k,2k}$, W_{u-k}, $Z_{k,k-u''-1,k,3k}$, and $Z_{k,u'',k,k}$ in (18) guarantee that each umpire visits the home venue of every team in T at least once. Therefore, S also satisfies constraint (ii).

Constraint (iii) is satisfied if all venues visited by a given umpire during any $n - d_1$ consecutive rounds are different. We will show that S satisfies (iii) for $d_1 = 0$ because this implies that it satisfies (iii) for any $0 \le d_1 \le n/2$. In terms of k, this is equivalent to saying that no umpire visits a venue more than once in any stretch of $2k$ rounds.

Sequences W, Y (see Lemma 4), and Z in (15)–(18) individually consist of different venues and, therefore, respect (iii). If two sequences in (15)–(18) are separated by $2k-1$ rounds or more, no venue that appears in both of them can appear twice within a stretch of $2k$ consecutive rounds. Therefore, they clearly satisfy (iii). When two sequences are separated by fewer than $2k-1$ rounds but consist of venues that are all different from each other, they satisfy (iii) as well. Hence, we focus on the sequences in (15)–(18) that are separated by fewer than $2k-1$ rounds and have venues in common.

During rounds $0, 2, \ldots, 2k-2$, umpires $0, \ldots, k/2-1$ travel through the venues of $Z_{k,u,k,3k}$ in (15), and umpires $k/2, \ldots, k-1$ travel through the venues of $Z_{k,u',k,2k}$ in (16). Each of these umpires visits the same sequence of venues again during rounds $3k-1, 3k+1, \ldots, 5k-3$. This means that a venue visited during rounds $0, 2, \ldots, 2k-2$ will only

Umpire	Rounds																						
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
	$(Z_{8,u,8,24} \oslash Y_{8,u,7,23,0})$															$Z_{8,u,8,8}$							
0	24	7	25	23	26	1	27	2	28	3	29	4	30	5	31	8	9	10	11	12	13	14	15
1	25	6	26	0	27	7	28	23	29	2	30	3	31	4	24	9	10	11	12	13	14	15	8
2	26	5	27	6	28	0	29	1	30	7	31	23	24	3	25	10	11	12	13	14	15	8	9
3	27	4	28	5	29	6	30	0	31	1	24	2	25	7	26	11	12	13	14	15	8	9	10
	$(Z_{8,u',8,16} \oslash Y_{8,u',15,31,8})$															$W_{u'}$							
4	16	15	17	31	18	9	19	10	20	11	21	12	22	13	23	0	1	2	3	4	5	6	7
5	17	14	18	8	19	15	20	31	21	10	22	11	23	12	16	1	2	3	4	5	6	7	0
6	18	13	19	14	20	8	21	9	22	15	23	31	16	11	17	2	3	4	5	6	7	0	1
7	19	12	20	13	21	14	22	8	23	9	16	10	17	15	18	3	4	5	6	7	0	1	2
	$(Z_{8,u',8,24} \oslash Y_{8,u-8,23,7,16})$															$Z_{8,u',8,8}$							
8	28	23	29	7	30	17	31	18	24	19	25	20	26	21	27	12	13	14	15	8	9	10	11
9	29	22	30	16	31	23	24	7	25	18	26	19	27	20	28	13	14	15	8	9	10	11	12
10	30	21	31	22	24	16	25	17	26	23	27	7	28	19	29	14	15	8	9	10	11	12	13
11	31	20	24	21	25	22	26	16	27	17	28	18	29	23	30	15	8	9	10	11	12	13	14
	$(Z_{8,u-8,8,16} \oslash Y_{8,u'',31,15,24})$															W_{u-8}							
12	20	31	21	15	22	25	23	26	16	27	17	28	18	29	19	4	5	6	7	0	1	2	3
13	21	30	22	24	23	31	16	15	17	26	18	27	19	28	20	5	6	7	0	1	2	3	4
14	22	29	23	30	16	24	17	25	18	31	19	15	20	27	21	6	7	0	1	2	3	4	5
15	23	28	16	29	17	30	18	24	19	25	20	26	21	31	22	7	0	1	2	3	4	5	6

Umpire	Rounds														
	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37
	$(Z_{8,u,8,24} \oslash Z_{7,u,7,16})$														
0	24	16	25	17	26	18	27	19	28	20	29	21	30	22	31
1	25	17	26	18	27	19	28	20	29	21	30	22	31	16	24
2	26	18	27	19	28	20	29	21	30	22	31	16	24	17	25
3	27	19	28	20	29	21	30	22	31	16	24	17	25	18	26
	$(Z_{8,u',8,16} \oslash Z_{7,u',7,24})$														
4	16	24	17	25	18	26	19	27	20	28	21	29	22	30	23
5	17	25	18	26	19	27	20	28	21	29	22	30	23	24	16
6	18	26	19	27	20	28	21	29	22	30	23	24	16	25	17
7	19	27	20	28	21	29	22	30	23	24	16	25	17	26	18
	$(Z_{8,15-u,8,16} \oslash Z_{7,u-8,7,0})$														
8	23	0	16	1	17	2	18	3	19	4	20	5	21	6	22
9	22	1	23	2	16	3	17	4	18	5	19	6	20	0	21
10	21	2	22	3	23	4	16	5	17	6	18	0	19	1	20
11	20	3	21	4	22	5	23	6	16	0	17	1	18	2	19
	$(Z_{8,7-u'',8,24} \oslash Z_{7,u'',7,8})$														
12	31	8	24	9	25	10	26	11	27	12	28	13	29	14	30
13	30	9	31	10	24	11	25	12	26	13	27	14	28	8	29
14	29	10	30	11	31	12	24	13	25	14	26	8	27	9	28
15	28	11	29	12	30	13	31	14	24	8	25	9	26	10	27

Umpire	Rounds																							
	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61
	W_u								$(Z_{8,u+4,8,8} \oslash Z_{8,u,8,16})$															
0	0	1	2	3	4	5	6	7	12	16	13	17	14	18	15	19	8	20	9	21	10	22	11	23
1	1	2	3	4	5	6	7	0	13	17	14	18	15	19	8	20	9	21	10	22	11	23	12	16
2	2	3	4	5	6	7	0	1	14	18	15	19	8	20	9	21	10	22	11	23	12	16	13	17
3	3	4	5	6	7	0	1	2	15	19	8	20	9	21	10	22	11	23	12	16	13	17	14	18
	$Z_{8,u',8,8}$								$(Z_{8,u',8,24} \oslash Z_{8,u,8,0})$															
4	8	9	10	11	12	13	14	15	24	4	25	5	26	6	27	7	28	0	29	1	30	2	31	3
5	9	10	11	12	13	14	15	8	25	5	26	6	27	7	28	0	29	1	30	2	31	3	24	4
6	10	11	12	13	14	15	8	9	26	6	27	7	28	0	29	1	30	2	31	3	24	4	25	5
7	11	12	13	14	15	8	9	10	27	7	28	0	29	1	30	2	31	3	24	4	25	5	26	6
	$Z_{8,u',8,8}$								$(Z_{8,u',8,24} \oslash Z_{8,u-8,8,0})$															
8	12	13	14	15	8	9	10	11	28	0	29	1	30	2	31	3	24	4	25	5	26	6	27	7
9	13	14	15	8	9	10	11	12	29	1	30	2	31	3	24	4	25	5	26	6	27	7	28	0
10	14	15	8	9	10	11	12	13	30	2	31	3	24	4	25	5	26	6	27	7	28	0	29	1
11	15	8	9	10	11	12	13	14	31	3	24	4	25	5	26	6	27	7	28	0	29	1	30	2
	W_{u-8}								$(Z_{8,u'',8,8} \oslash Z_{8,u-8,8,16})$															
12	4	5	6	7	0	1	2	3	8	20	9	21	10	22	11	23	12	16	13	17	14	18	15	19
13	5	6	7	0	1	2	3	4	9	21	10	22	11	23	12	16	13	17	14	18	15	19	8	20
14	6	7	0	1	2	3	4	5	10	22	11	23	12	16	13	17	14	18	15	19	8	20	9	21
15	7	0	1	2	3	4	5	6	11	23	12	16	13	17	14	18	15	19	8	20	9	21	10	22

Fig. 4. Solution S when $k = 8$ and $C = 0, 1, 2, 3, 4, 5, 6, 7, 0$. Total distance traveled = 864.

Table 1
Assignment of home venues in T to the umpires in S.

Rounds	Home venues in T	Umpires	Subsequence in S
$0, 2, \ldots, 2k-2$	$\overline{P}_{2k,0}[0, k-1]$	$0 \ldots k/2-1$ $k/2 \ldots k-1$ $k \ldots 3k/2-1$ $3k/2 \ldots 2k-1$	$Z_{k,u,k,3k}$ $Z_{k,u',k,2k}$ $Z_{k,u',k,3k}$ $Z_{k,u-k,k,2k}$
$1, 3, \ldots, 2k-3$	$\overline{U}_{k,0}$ $\overline{U}_{k,k}$ $\overline{U}_{k,2k}$ $\overline{U}_{k,3k}$	$0 \ldots k/2-1$ $k/2 \ldots k-1$ $k \ldots 3k/2-1$ $3k/2 \ldots 2k-1$	$Y_{k,u,k-1,3k-1,0}$ $Y_{k,u',2k-1,4k-1,k}$ $Y_{k,u-k,3k-1,k-1,2k}$ $Y_{k,u'',4k-1,2k-1,3k}$
$2k-1, 2k, \ldots, 3k-2$	$P_{2k,0}[0, k-1]$	$0 \ldots k/2-1$ $k/2 \ldots k-1$ $k \ldots 3k/2-1$ $3k/2 \ldots 2k-1$	$Z_{k,u,k,k}$ $W_{u'}$ $Z_{k,u',k,k}$ W_{u-k}
$3k-1, 3k+1, \ldots, 5k-3$	$\overline{P}_{2k,0}[k, 2k-1]$	$0 \ldots k/2-1$ $k/2 \ldots k-1$ $k \ldots 3k/2-1$ $3k/2 \ldots 2k-1$	$Z_{k,u,k,3k}$ $Z_{k,u',k,2k}$ $Z_{k,2k-u-1,k,2k}$ $Z_{k,k-u''-1,k,3k}$
$3k, 3k+2, \ldots, 5k-4$	$U_{k,0}$ $U_{k,k}$ $U_{k,2k}$ $U_{k,3k}$	$k \ldots 3k/2-1$ $3k/2 \ldots 2k-1$ $0 \ldots k/2-1$ $k/2 \ldots k-1$	$Z_{k-1,u-k,k-1,0}$ $Z_{k-1,u'',k-1,k}$ $Z_{k-1,u,k-1,2k}$ $Z_{k-1,u',k-1,3k}$
$5k-2, 5k-1, \ldots, 6k-3$	$P_{2k,0}[k, 2k-1]$	$0 \ldots k/2-1$ $k/2 \ldots k-1$ $k \ldots 3k/2-1$ $3k/2 \ldots 2k-1$	W_u $Z_{k,u',k,k}$ $Z_{k,u',k,k}$ W_{u-k}
$6k-2, 6k, \ldots, 8k-4$	$\overline{P}_{k,0}$ $\overline{P}_{k,2k}$	$0 \ldots k/2-1$ $3k/2 \ldots 2k-1$ $k/2 \ldots k-1$ $k \ldots 3k/2-1$	$Z_{k,u+k/2,k,k}$ $Z_{k,u'',k,k}$ $Z_{k,u',k,3k}$ $Z_{k,u',k,3k}$
$6k-1, 6k+1, \ldots, 8k-3$	$P_{k,0}$ $P_{k,2k}$	$k/2 \ldots k-1$ $k \ldots 3k/2-1$ $0 \ldots k/2-1$ $3k/2 \ldots 2k-1$	$Z_{k,u,k,0}$ $Z_{k,u-k,k,0}$ $Z_{k,u,k,2k}$ $Z_{k,u-k,k,2k}$

be visited again by the same umpire after $3k-1$ rounds. Analogously, the venues visited by all of the umpires in rounds $3k, 3k+2, \ldots, 5k-4$ (sequences $Z_{k-1,u,k-1,2k}$, $Z_{k-1,u',k-1,3k}$, $Z_{k-1,u-k,k-1,0}$, and $Z_{k-1,u'',k-1,k}$ in (15)–(18)) are visited again by umpires $0, \ldots, k/2-1$ and $k-1, \ldots, 3k/2$ in rounds $6k-1, 6k+1, \ldots, 8k-3$ (sequences $Z_{k,u,k,2k}$ and $Z_{k,u-k,k,0}$ in (15) and (17)), and by umpires $k/2, \ldots, k-1$ and $3k/2, \ldots, 2k-1$ in rounds $6k-2, 6k, \ldots, 8k-4$ (sequences $Z_{k,u',k,3k}$ and $Z_{k,u'',k,k}$ in (16) and (18)). By Lemma 3, the i-th venue in the sequences visited by an umpire in rounds $3k, 3k+2, \ldots, 5k-4$ appears exactly once in the sequence visited by the same umpire in rounds $6k-2, 6k, \ldots, 8k-4$ or $6k-1, 6k+1, \ldots, 8k-3$, being the i-th or $(i+1)$-th venue of the latter two sequences. Hence, the venues visited in rounds $3k, 3k+2, \ldots, 5k-4$ are only visited again by the same umpire after no fewer than $3k-2$ rounds.

We now look at sequences $Y_{k,u-k,3k-1,k-1,2k}$ and $Z_{k,2k-u-1,k,2k}$ in (17), which are traversed by umpires $k, \ldots, 3k/2-1$ in rounds $1, 3, \ldots, 2k-3$ and $3k-1, 3k+1, \ldots, 5k-3$, respectively. Using (12)–(14) to generate $Y_{k,u-k,3k-1,k-1,2k}$ yields

$$Y_{k,u-k,3k-1,k-1,2k} = (3k-1, k-1) \oplus Z_{k-1,u,3k-2u-3,2k}, \quad \text{if } u = k, \tag{19}$$

$$Y_{k,u-k,3k-1,k-1,2k} = Z_{k-1,2k-u-1,2u-2k,2k} \oplus (3k-1, k-1) \oplus Z_{k-1,u,3k-2u-3,2k}, \quad \text{if } k+1 \le u \le 3k/2-2, \tag{20}$$

$$Y_{k,u-k,3k-1,k-1,2k} = Z_{k-1,2k-u-1,2u-2k,2k} \oplus (3k-1), \quad \text{if } u = 3k/2-1. \tag{21}$$

First, note that venue $k-1$ in $Y_{k,u-k,3k-1,k-1,2k}$ does not appear in $Z_{k,2k-u-1,k,2k}$. Venue $3k-1$ appears in the $(2(u-k)+1)$-th position in the $Y_{k,u-k,3k-1,k-1,2k}$ sequence assigned to umpire u. Therefore, u visits this venue in round $4(u-k)+1$. Given the definition of $Z_{k,2k-u-1,k,2k}$, consider the equation $2k + ((2k-u-1+i) \bmod k) = 3k-1$ for $0 \le i \le k-1$. Since $0 \le 2k-u-1 \le k-1$ when $k \le u \le 3k/2-1$, this equation simplifies to $(2k-u-1+i) = k-1$. Hence, we know that venue $3k-1$ occupies the $((u-k)+1)$-th position in the $Z_{k,2k-u-1,k,2k}$ sequence assigned to umpire u, which implies that u visits $3k-1$ in round $3k-1+2(u-k) = 2u+k-1$. This implies that venue $3k-1$ is visited again by the same umpire u after $2u+k-1-(4(u-k)+1) = 5k-2u-2 \ge 2k$ rounds when $4 \le k \le u \le 3k/2-1$.

Next, note that $Z_{k,2k-u-1,k,2k} = Z_{k,2k-u-1,2u-2k+1,2k} \oplus Z_{k,u,3k-2u-2,2k} \oplus Z_{k,3k-u-2,1,2k}$. Umpires $k \le u \le 3k/2-2$ visit the venues of $Z_{k-1,u,3k-2u-3,2k}$ in (19) and (20) during rounds $4(u-k)+5, 4(u-k)+7, \ldots, 2k-3$, and visit these venues again in $Z_{k,u,3k-2u-2,2k}$ (a subsequence of $Z_{k,2k-u-1,k,2k}$) during rounds $3k-1+2(2u-2k+1) = 4u-k+1, 4u-k+3, \ldots, 5k-5$.

Umpires $k+1 \le u \le 3k/2-1$ visit the venues of $Z_{k-1,2k-u-1,2u-2k,2k}$ in (20) and (21) during rounds $1, 3, \dots, 4(u-k)-1$, and visit these venues again in $Z_{k,2k-u-1,2u-2k+1,2k}$ (a subsequence of $Z_{k,2k-u-1,k,2k}$) during rounds $3k-1, 3k+1, \dots, 4u-k-1$. Applying Lemma 3 to the two cases above, and considering sequences $Z_{k-1,u,3k-2u-3,2k}$ and $Z_{k,u,3k-2u-2,2k}$, as well as $Z_{k-1,2k-u-1,2u-2k,2k}$ and $Z_{k,2k-u-1,2u-2k+1,2k}$, we conclude that these umpires only return to the venues visited in rounds $1, 3, \dots, 2k-3$ after at least $3k-4$ rounds.

The reasoning from the previous paragraph can be applied to sequences $Y_{k,u'',4k-1,2k-1,3k}$ and $Z_{k,k-u''-1,k,3k}$ in (18), because they are equivalent to $Y_{k,u-k,3k-1,k-1,2k}$ and $Z_{k,2k-u-1,k,2k}$, respectively, with the venue IDs increased by k. Therefore, we conclude that all sequences in (15)–(18) that can have venues in common and are separated by fewer than $2k-1$ rounds satisfy (iii), which implies that S is feasible.

All of the venues in W sequences have IDs in the interval $[0, k-1]$, and they appear in the same order that they do in the hamiltonian circuit C. Therefore, any two consecutive venues in a W sequence correspond to two adjacent vertices in G. As a consequence, the umpires that traverse these sequences go on $k-1$ 0-trips. In S, umpires $k/2, \dots, k-1$ and $3k/2, \dots, 2k-1$ traverse sequences $W_{u'}$ and W_{u-k}, respectively, in rounds $2k-1, \dots, 3k-2$, whereas umpires $0, \dots, k/2-1$ and $3k/2, \dots, 2k-1$ traverse sequences W_u and W_{u-k}, respectively, in rounds $5k-2, \dots, 6k-3$. Hence, these umpires go on exactly $2k(k-1)$ 0-trips. Because all of the remaining trips are 1-trips, the total distance traveled by the $2k$ umpires in S is equal to $n(4n-3) - 2k(k-1)$. □

Finding a feasible solution with a well-defined structure becomes very difficult when constraint (iv) is enforced with $d_2 < \lfloor n/2 \rfloor - 1$. Therefore, in the proof of $\mathcal{NP}$-completeness (Theorem 2, (⇒) direction), we turn off constraint (iv) by setting $d_2 = \lfloor n/2 \rfloor - 1$ for the solution defined by (15)–(18) to be feasible. To see why, consider Figs. 3 and 4 together. The sequences of venues corresponding to the hamiltonian circuit C defined with W in (15), (16), and (18) do not always result in a solution satisfying (iv) if $d_2 < \lfloor n/2 \rfloor - 1$.

Corollary 1. *The decision version of the TUP with $d_1 \le n/2$ and $d_2 = \lfloor n/2 \rfloor - 1$ is an $\mathcal{NP}$-complete problem.*

Proof. This decision version of the TUP is clearly in $\mathcal{NP}$. Instance $I(G)$ of Definition 2 can be created in $O(k^2)$ time, and Theorems 1 and 2 show that the polynomial-time reduction of an $\mathcal{NP}$-complete problem to this decision version of the TUP is correct. □

4. Conclusions and future work

Before this work, the complexity of the TUP was still open. We provide a formal proof that the decision version of the TUP is a computationally difficult problem. This result is not surprising given all the empirical evidence gathered from several papers that have attempted to tackle increasingly larger instances of this problem since it was first introduced in [2]. We hope this work will motivate other researchers to further advance the theory surrounding this and other sports scheduling problems, as well as encourage the development of new computational approaches to deal with them.

As future work, we intend to investigate an extension of the proof presented in this paper to show that the TUP remains $\mathcal{NP}$-complete even when $d_1 = d_2 = 0$. We believe this to be true, but expect the proof to be significantly more elaborate. We also suspect that the problem of deciding whether or not a given TUP instance is feasible is $\mathcal{NP}$-complete, as practical experience suggests. This is another research direction we are currently pursuing.

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